Modal Parameter Identification from Ambient Response

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Introduction

MODAL parameter identification from ambient vibration data has gained considerable attention in recent years. The ambient vibration survey for determining dynamic characteristics of engineering structures is a valuable tool for practical structural health monitoring.[1,2] Ibrahim et al.[3] applied the random decrement technique coupled with a time-domain parameter identification method (ITD)[4] to process ambient vibration data. Although the random decrement technique serves as an alternative method for estimating the autocorrelation and cross-correlation functions, it is based on an intuitive theory and does not yet have a sound mathematical basis for general cases.[5] James et al.[6,7] developed the so-called natural excitation technique using the cross-correlation technique coupled with time-domain parameter extraction. This concept becomes a very powerful tool for the analysis of structures under ambient vibration.

In the present Note, a theoretical justification of the cross-correlation technique is presented for a linear, complex-mode system excited by white-noise random inputs. By treating the sample correlations of measured response as output from free vibration decay, a time-domain modal identification method, such as the ITD method,[4] can then be employed to extract (complex) modal parameters of a structure. Numerical simulations using a seven-degree-of-freedom (DOF) linear system, which has two pairs of closely spaced modes,[8] were performed to verify the correlation technique and to investigate the effectiveness of the coupled ITD method. The robustness of the proposed approach is also examined in identifying modal parameters of a linear system from noisy measurements.

Theoretical Development of Cross-Correlation Functions

The equations of motion of a discrete linear system can be expressed in state space as

$$[A]\dot{\boldsymbol{X}} + [B]\boldsymbol{X} = \boldsymbol{F} \tag{1}$$

where

$$[A] = \begin{bmatrix} C & M \\ M & O \end{bmatrix}, \qquad [B] = \begin{bmatrix} K & O \\ O & -M \end{bmatrix}$$
$$\boldsymbol{X} = \begin{Bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{Bmatrix}, \qquad \boldsymbol{F} = \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{Bmatrix} \tag{2}$$

In Eq. (2), $[M]$, $[C]$, and $[K]$ are the mass, damping, and stiffness matrices, respectively; $\boldsymbol{x}(t)$ is the displacement vector; and $\boldsymbol{f}(t)$ is the force vector. Introduce the transformation

$$\boldsymbol{X}(t) = [\Psi]\boldsymbol{q}(t) = \sum_{r=1}^{2m} \boldsymbol{\psi}_r q_r(t) \tag{3}$$

where $[\Psi]$ denotes the complex modal matrix, $\boldsymbol{q}(t)$ the vector of modal coordinates, and $\boldsymbol{\psi}_r$ the vector of the rth mode shape. Premultiplying Eq. (1) by $[\Psi]^T$ we can transform Eq. (1) from physical coordinates into modal coordinates as follows:

$$\dot{q}_r(t) - \lambda_r q_r(t) = (1/a_r)\boldsymbol{\psi}_r^T \boldsymbol{F}(t), \qquad r = 1 \sim 2m \tag{4}$$

where we used the orthogonality of mode shapes

$$[\Psi]^T[A][\Psi] = [a], \qquad [\Psi]^T[B][\Psi] = [b] \tag{5}$$

so that $[a]$, $[b]$ are diagonal matrices with elements a_r and b_r, $\lambda_r = -(b_r/a_r)$, and $r = 1 \sim 2m$. If we assume that the system is initially at rest, then the solution to Eq. (4) can be found, and the response at the ith DOF due to the input at the kth DOF can be derived as

$$x_{ik}(t) = \sum_{r=1}^{2m} \frac{\psi_{ir}\psi_{kr}}{a_r} \int_{-\infty}^{t} e^{\lambda_r(t-\tau)} f_k(\tau)\,\mathrm{d}\tau \tag{6}$$

where ψ_{ir} denotes the ith component of the rth mode shape.

Define the cross-correlation function $R_{ijk}(T)$ between two stationary response signals $x_{ik}(t)$ and $x_{jk}(t)$ as

$$R_{ijk}(T) = E[x_{ik}(t+T)x_{jk}(t)] \tag{7}$$

Substituting Eq. (6) into Eq. (7) and assuming that $f_k(t)$ is white noise, i.e., $E[f_k(\tau)f_k(\sigma)] = \alpha_k\delta(\tau-\sigma)$, where α_k is a constant and $\delta(t)$ is the Dirac delta function, we can derive

$$R_{ijk}(T) = \sum_{r=1}^{2m}\sum_{s=1}^{2m} \frac{-\alpha_k\psi_{ir}\psi_{kr}\psi_{js}\psi_{ks}}{a_r a_s(\lambda_r+\lambda_s)} e^{\lambda_r T} \tag{8}$$

where we note that the real parts of λ_r are negative for damped stable systems. To find the cross-correlation function due to all of the inputs $f_k(t)$, $k = 1 \sim N$, which are assumed to be uncorrelated with one another, we use Eq. (8) and sum over all of the input locations to get

$$R_{ij}(T) = \sum_{r=1}^{2m} A_{jr}\psi_{ir} e^{\lambda_r T} \tag{9}$$

where A_{jr} is just a (complex) constant defined by

$$A_{jr} = \sum_{s=1}^{2m}\sum_{k=1}^{N} \frac{-\alpha_k\psi_{kr}\psi_{js}\psi_{ks}}{a_r a_s(\lambda_r+\lambda_s)} \tag{10}$$

The preceding result shows that the cross-correlation function in Eq. (9) is a sum of complex exponential functions that is of the same form as the free vibration decay or the impulse response of the original system.[4] Thus, cross-correlation functions of responses can be used as free vibration decay or as an impulse response in time-domain modal extraction schemes so that measurement of white-noise inputs can be avoided. It is remarkable that the term $A_{jr}\psi_{ir}$ in the cross-correlation function of Eq. (9) will be identified as the mode-shape component. To eliminate the A_{jr} term and retain the true mode-shape component ψ_{ir}, all of the measured channels are correlated against a common reference channel x_j. The identified components then all possess the common A_{jr} component, which can be normalized out to obtain the ψ_{ir}.

ITD Method

The Ibrahim and Mikulcik[4] ITD method uses free decay responses of a structure under testing to identify its modal parameters in complex form. In the method, a system matrix $[A]$ is defined so that it satisfies

$$[A]_{(n\times n)}[X]_{(n\times q)} = [Y]_{(n\times q)} \tag{11}$$

where $[X]$ and $[Y]$ are measured response data matrices containing elements $x_{ij} \equiv x_i(t_j)$ and $y_{ij} \equiv x_i(t_j+\Delta t)$, respectively, where $i = 1 \sim n$, $j = 1 \sim q$, and Δt is a time delay for physical measurement. Generally, the number of sampling points q is chosen larger than the number of measurement channels n. Therefore, matrix $[A]$ can be estimated by using the least-squares method as

$$[A] = [Y][X]^T([X][X]^T)^{-1} \tag{12}$$

It can be shown that an eigenvalue problem can be formulated as[4]

$$[A]\phi_r = \rho_r\phi_r \tag{13}$$

where $\rho_r = e^{\lambda_r\Delta t}$ is the rth eigenvalue of $[A]$ and λ_r are the characteristic values of the original vibrating system, which can thus be obtained from ρ_r. It can also be shown that the rth eigenvector ϕ_r corresponding to the eigenvalue ρ_r of the system matrix $[A]$ is just the rth-mode shape of the original vibrating system.

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Numerical Simulation

Consider a linear system having seven DOF as shown in Fig. 1 (Ref. 8). Assume that each mass is 1 kg and the spring constants are $k_1 = 10$ kN/m and $k_2 = 20$ kN/m. The system damping matrix is assumed to be $[C] = 0.2[M] + 0.0003[K]$, so that the system has two pairs of closely spaced modes whose frequencies are around 28 and 40 Hz, respectively. Although we assumed proportional damping, which implies classical normal modes, the modal parameter identification scheme with ITD still treats the system as having complex modes. The main focus here is to examine the effectiveness of the proposed method in identifying closely spaced modes, which is a difficult situation for methods based on other techniques, such as modal sweep.[1]

The excitation signal is generated using the spectrum approximation method[9] as a zero-mean band-pass noise, whose standard deviation is 2.24 ms^{-2} with a frequency range from 0 to 250 Hz. The sampling rate is 500 Hz, and the sampling period is 8.19 s. The white-noise excitation is assumed to act on the first DOF. To account for measurement noise, 20% random noise is added to the response signal. Applying the aforementioned correlation method along with the Ibrahim and Mikulcik[4] time-domain method, we conduct modal parameter identification analysis for the seven-DOF linear system.

The results of modal parameter identification are summarized in Table 1, which shows that the errors in frequencies are less than 3% and the maximum error in damping ratios is about 30%. The identified mode shapes are compared with exact values in Fig. 2, where only those of the two pairs of close modes (modes 3, 4 and 5, 6) are displayed. It can be seen that good results have been obtained from response data contaminated with 20% noise for the system having two pairs of closely spaced modes. The effectiveness and robustness of the proposed approach, which combines the ITD method and the correlation technique, are verified. The method is valid in identifying modes with identical (close) frequencies but with different damping ratios, inasmuch as the corresponding eigenvalues of the system matrix $[A]$ are then different. The method is robust to measurement noise because that random noise is partly eliminated through the evaluation of correlation functions. For general (nonwhite) excitation, techniques need to be developed to sort out dynamic properties of the structures from those of the excitations.[10]

Table 1 Results of modal parameter identification from simulated data with 20% noise

	Natural frequencies, Hz		Damping ratio, %	
Mode	Exact	ITD	Exact	ITD
1	13.387	13.411	1.38	1.86
2	22.848	23.223	2.22	2.57
3	28.174	28.361	2.71	2.33
4	28.868	29.253	2.78	3.74
5	40.145	39.621	3.82	3.24
6	40.386	40.291	3.94	4.89
7	46.900	45.660	4.45	4.23

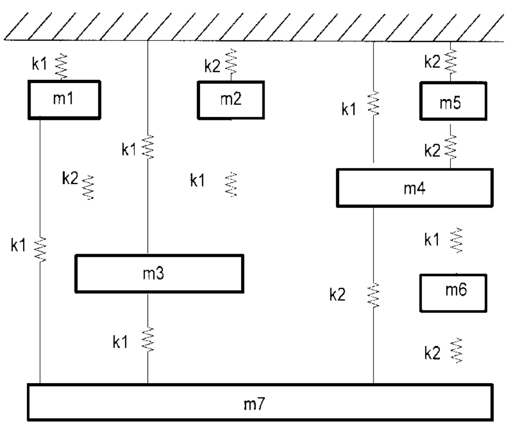

Fig. 1 Linear system with seven DOF and with two pairs of close modes.

Conclusions

An effective identification method is developed for determination of the modal parameters of a structure from its measured ambient response data. The method combines the Ibrahim et al. time-domain method and the correlation technique and applies to general, linear structures subject to white-noise excitation. Simulation studies demonstrated that the method is effective in identifying complex modes even with close frequencies and is robust to measurement noise. Further investigation could be made of identification from ambient response of systems subject to nonwhite excitation.

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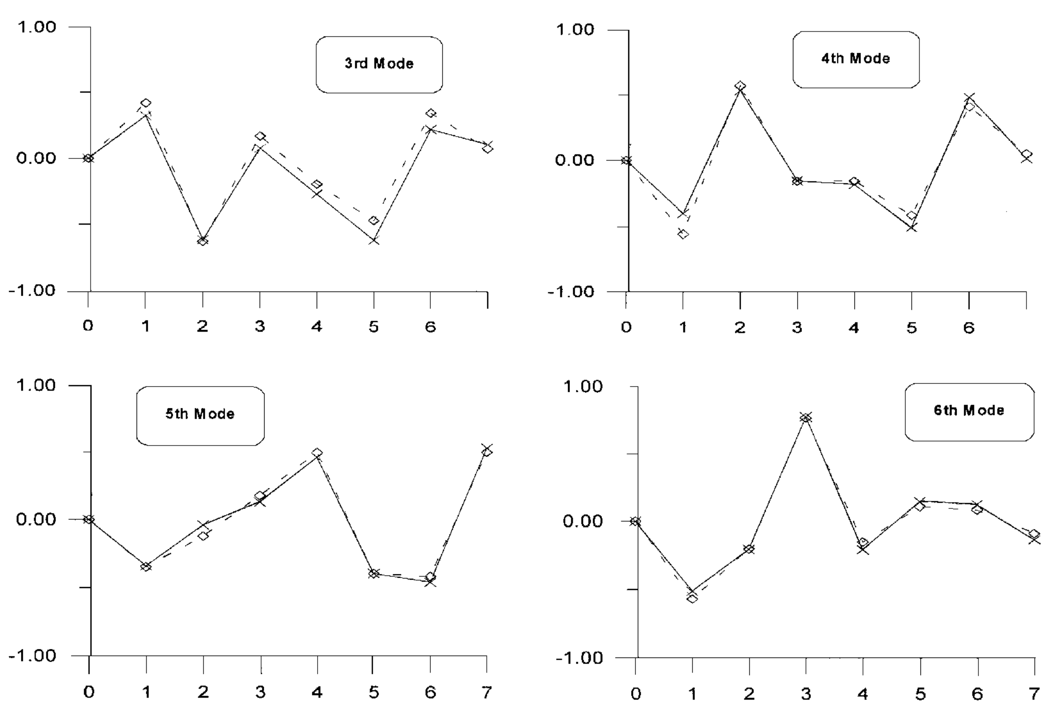

Fig. 2 Comparison of identified mode shapes (◇) with exact values (×) for the two pairs of closely spaced modes.

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Interpretation of a Linear Solver Based on Davidson's Method

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Introduction

DAVIDSON[1] originally developed an algorithm for the solution of an eigenvalue problem, which was modified and converted into a linear solver. However, no existing documentation was available on the development of this algorithm. The linear solver is an extremely efficient algorithm and is capable of solving enormous systems of equations in a very short time span. A system of 2000 equations was solved in less than 5 min, taking approximately 40 iterations.[2] This Note provides a geometric interpretation of the method and is presented as a development of the algorithm.

The Panel Method Ames Research Center program developed by NASA Ames Research Center contained a subroutine for a linear solver based on Davidson's method. The source code for this subroutine was extracted and studied, and an algorithm for the method was derived.

Solution of Linear Systems of Equations

The following nomenclature is used: Given a system of linear equations, the coefficient matrix $\boldsymbol{A}$, the constant vector $\boldsymbol{b}$, and the solution vector $\boldsymbol{x}$ can be expressed as

$$\boldsymbol{Ax} = \boldsymbol{b} \tag{1}$$

In an iterative process, the kth iteration will be written as $\boldsymbol{x}^{(k)}$. The residual vector is

$$\boldsymbol{r}^{(k)} = \boldsymbol{b} - \boldsymbol{Ax}^{(k)} \qquad \text{for} \qquad k = 1, 2, \ldots, n \tag{2}$$

When $|\boldsymbol{r}| < \varepsilon$, convergence is assumed, where ε is the convergence criterion. Traditional solvers keep altering approximations to the solution until Eq. (2) is less than a desired tolerance ε. Another solution method involves an optimization problem. When the coefficient matrix is symmetric, the location of the absolute maximum or minimum of the associated quadratic form is equivalent to solving the linear system of equations.[3,4]

The quadratic form of Eq. (1) can be expressed as follows:

$$F(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{Ax})/2 - (\boldsymbol{b}, \boldsymbol{x}) \tag{3}$$

At the end of this Note, a comparison is made between the performances of the conjugate gradient (CG) method and Davidson's method using symmetric matrices. The values of the quadratic function obtained by both methods are listed along with the number of iterations taken.

Description of Davidson's Method

Davidson's method has been found to outperform the CG method. In addition, symmetry is not a prerequisite. In the CG method, asymmetric matrices must be preconditioned. The manner in which Davidson's method searches for the solution is primarily responsible for its efficiency. The CG method always conducts its search in a two-dimensional plane spanned by two orthogonal vectors. It then searches for an approximation to the solution in that plane. At the next iteration, a new search is conducted in a plane orthogonal to the previous one. This process continues until the convergence criterion has been satisfied.

Davidson's method, however, takes this idea one step further and conducts its search in an increasing orthonormal base. At the first iteration (Fig. 1), the method will normalize the initial guess $\boldsymbol{u}$ and search for the best approximation to the solution that is a scalar multiple of the normalized initial guess vector $\boldsymbol{v}^{(0)}$. At the next iteration, a unit vector $\boldsymbol{v}^{(1)}$, orthogonal to the previous vector, is calculated (using a variation of the Gram–Schmidt technique[5]). The next approximation to the solution is sought in the plane spanned by $\boldsymbol{v}^{(0)}$ and $\boldsymbol{v}^{(1)}$ (Fig. 2). A new vector $\boldsymbol{v}^{(2)}$ is then calculated that is orthogonal to both $\boldsymbol{v}^{(0)}$ and $\boldsymbol{v}^{(1)}$. The search is now conducted in $\boldsymbol{R}^3$. The span of the space in which the search is conducted increases with subsequent iterations.

Another major difference between the two methods is the manner in which the residual vector $\boldsymbol{r}$ is used to influence the search. In

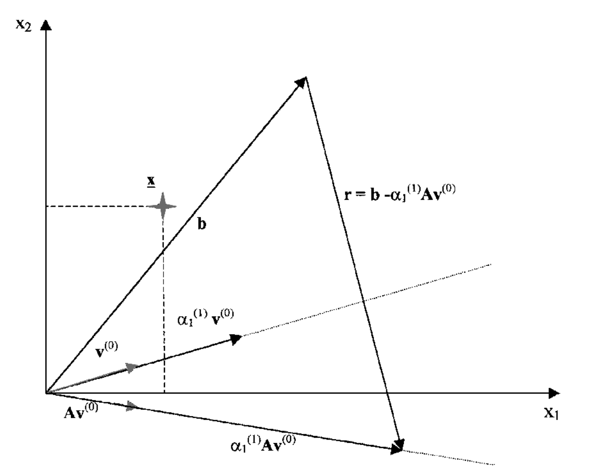

Fig. 1 First iteration of Davidson's method for a system of two equations; $\boldsymbol{v}^{(0)}$ is the normalized initial guess $\boldsymbol{u}$ (not shown).

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